A NOTE ON THE RIGHT IDEAL STATEMENTS FOR NON-ASSOCIATIVE RINGS

Ali KARAKUŞ

Kilis 7 Aralık University, Department of Mathematics

ORCID: 0000-0002-8483-0137

ABSTRACT

In non-associative ring theory the lattice of right ideals has not proven nearly as useful a tool as in the associative theory. A partial reason for this is that given an element x in a non-associative ring R, xR need not be a right ideal of R. In this paper we wish to investigate, under certain limitations, some statements for non-associative rings R that imply xR is a right ideal for every x in the ring R.

KEY WORDS: Right ideal, Rings, Non-associative rings, Lattice.

1. INTRODUCTION

Non-associative rings arise when we relax one of the defining properties of classical ring theory: associativity of multiplication. In a standard ring, multiplication satisfies

$$(ab)c = a(bc)(ab)c = a(bc)(ab)c = a(bc)$$

for all elements. However, in many algebraic systems that appear naturally in mathematics and physics, this condition is too restrictive. By dropping associativity while retaining the additive structure of a group (usually abelian) and distributivity of multiplication over addition, we obtain what is called a non-associative ring.[1][2]

Formally, a non-associative ring is a set equipped with two binary operations, addition and multiplication, such that:

The set forms an abelian group under addition.

Multiplication is distributive over addition from both sides.

Multiplication is not required to be associative.

Because associativity is no longer guaranteed, new phenomena emerge. For instance, the order and grouping of terms in a product matter, so expressions must often be handled with greater care. This leads to the study of additional identities that measure how far a structure deviates from associativity, such as the associator

$$(a,b,c) = (ab)c - a(bc)(a,b,c) = (ab)c - a(bc)(a,b,c) = (ab)c - a(bc).$$

Important classes of non-associative rings include alternative rings, Jordan rings, and Lie rings. Each of these satisfies weaker forms of associativity or replaces it with other structural identities. For example, Lie rings play a central role in algebra and theoretical physics, particularly in the study of symmetries, while Jordan rings arise in quantum mechanics and projective geometry.[3][4]

Non-associative ring theory broadens the scope of algebra by encompassing structures that better model certain geometric, physical, and computational systems. Although more complex than associative rings, they provide a rich framework for exploring generalized multiplication and its consequences.

If R has an identity element, the latter property is equivalent to the statement: For every $x \in R$,

$$(xR)R = x(RR) \tag{1.1}$$

We will attempt to show why (1.1) is not a sufficiently strong assumption and why the following statements(S), which are strengthenings of (1.1), are reasonable to assume:

S1. For all right ideals L_1 and L_2 of R and for every $x \in R$, $(xL_1)L_2 = x(L_1L_2)$

S2. For every right ideal L of R and for all x and y in R, $x(Ly) = (xL)y$.

S3. For every right ideal L of R and for all x and y in R, $(xy)L = x(yL)$.

One can readily observe that either of S2 or S3 implies S1. [5][6][7]

2. MAIN RESULT

DEFINITION 2.1. By a ring R we mean an ordered triple $\langle R,+,\cdot\rangle$ where $\langle R,+\rangle$ is an Abelian group and "$\cdot$" is a binary operation on R such that for all $a,b,c \in R$

i) $a\cdot(b+c) = (a\cdot b)+(a\cdot c)$,

ii) $(a+b)\cdot c = (a\cdot c)+(b\cdot c)$.

As usual, we adopt the convention of using juxtaposition instead of "$\cdot$".

DEFINITION 2.2. If R is a ring we define $(\cdot,\cdot,\cdot): R\times R\times R \to R$, called the associator of R, by $(x,y,z) = (xy)z - x(yz)$.

DEFINITION 2.3. Suppose that R is a ring and $x \in R$. We define $x^1 = x$, $x^{n+1} = x^n x$ for $n \geq 1$. If R has identity 1 and $x \neq 0, x^0 = 1$.

We say that a non-zero element $x \in R$ is nilpotent with nilpotent exponent n if all possible products of x times itself n times yields 0 and some product of x times itself $(n-1)$ times is not zero.

DEFINITION 2.4. Suppose A and B are subsets of a ring R. We define

i) $AB = \{ab : a \in A, b \in B\}$,

ii) $A \circ B = \left\{ \sum c : c \in AB,\ finite\ sums \right\}$,

iii) $A^1 = A,\ A^{n+1} = A \circ A^n\ \ for\ n \geq 1$,

iv) $\ell - ann(A) = \left\{ r \in R : rA = \{0\} \right\}$.

We say that A is nil if every element of A is nilpotent; we say that A is nilpotent with nilpotent exponent n if all possible products of any n elements of A yields 0 and, for $A \neq \{0\}$ some product of $(n-1)$ many elements from A is not 0.

DEFINITION 2.5. Suppose that R is a ring. We say that R is a right (left) division ring if for every $0 \neq a$ and b in R, there is an x in R such that $xa = b(ax = b)$. If R is both a left and right-division ring we call R a division ring. We call R a unique (left-, right-) division ring if R is a (left-, right-) division ring with no proper zero divisors.

NOTATION 2.6. We will use the notation $\sum R_i$ for the internal sum, not necessarily direct, of the groups $\langle R_i, + \rangle$, $\sum \oplus R_i$ for the internal direct sum, and $+R_i$ for the external direct sum. Throughout this paper when we say that R satisfies $D.C.C.(A.C.C)$ we mean that R satisfies the descending (ascending) chain condition on its right ideals.

2.2. Motivation

We now list some theorems and examples, without proofs, for motivation:

THEOREM 2.7. Any right-division ring satisfies S1.

THEOREM 2.8 (Neumann, [3]). Any algebra can be embedded in a (right -) division algebra with identity.

THEOREM 2.9. If D is a uniplle right -division ring, then D satisfies S3.

THEOREM 2.10 (Neumann, [3]). If A is an algebra with identity 1 which has no proper zero divisors, then A can be embedded in a unique (right -) division algebra with identity 1.

EXAMPLE 2.11. Let $Z_P \subseteq F \subseteq K$ where K is the algebraic closure of the field F. Define the ring R by $R = Z_P + F$ with the multiplication

$$(a,b) * (c,d) = (ac, ad + bc + b \circ d), \tag{2.1}$$

where "$\circ$" denotes a multiplication on F.

Case 1. If we take "$\circ$" defined by

$$b \circ d = bd^p,$$

then $\langle F, +, \circ \rangle$ satisfies S2 and S3, where R under "$*$" is the usual manner of adjoining an identity to the ring $\langle F, +, \circ \rangle$ but $\langle R, +, * \rangle$ does not even satisfy S1.

Case 2. Now take $F = K$ and define "$\circ$" by

$$b \circ d = b^p d^p.$$

Then $\langle R,+,*\rangle$ satisfies the following:

i) R has an identity.
ii) R has only one proper right ideal, $\{0\}+K$.
iii) R has no non-zero nil right ideals.
iv) R is not a ring direct sum of any of its proper ideals.
v) R satisfies S1 and S2.
vi) R does not satisfy S3.

Remark. If R is a ring direct sum (product) of rings satisfying S1, then R satisfies S1, $1 \le i \le 3$.

Remark. It is possible to show, using the construction given in [3], that there exist unique right-division rings D with proper right ideals which do not satisfy S2 and which do not satisfy the dual to S1 where "right ideal" is replaced by "left ideal".

The following example gives some indication of the weakness of structure possible and the lack of duality in the right and left ideal structures for rings which satisfy (1.1) or the dual of S1.

The following hold for R:

i) e_1+e_2 is the identity for R.
ii) For every $x\in R$, $(xR)R=x(RR)$.
iii) R has no proper left ideals.
iv) For all left ideals R_1 and R_2, of R and every $x \in R$,
$(xR_1)R_2 = x(R_1R_2)$.
v) R is not a left or right-division algebra.
vi) R does not satisfy S1, S2, or S3.

2.3. Radical Theory

PROPOSITION 2.13. Suppose R satisjies S1. Then for every right ideal L of R and every $x \in R$, xL is a right ideal of R. If R satisfies S3, then $\ell-$ann (T) is a right ideal of R for every set $T \subseteq R$.

Proof. Suppose L is a right ideal of R and $x \in R$. CIearly xL is closed under addition. By S1, $(xL)R=x(RL)\subset xL$. Suppose that R satisfies S3 and

$T \subseteq R$. Clearly ℓ -ann (T) is an additive subgroup of R. Suppose $s \in R$, $r\in \ell-ann(T)$, and $t \in T$. By S3, there is an $\bar{s}\in R$ such that $(sr)t=\bar{s}(rt)=0$. Thus, $(sr)T=\{0\}$ and $\ell-ann(T)$ is a right ideal of R.

PROPOSITION 2.14. Suppose R satisfies S1. For any positive integer n and any right ideal L of R, L^n is a right ideal of R.

Proof. The statement is clear for $n=1$. Suppose $n>1$. By Definition 2.4 (iii) and S1,

$$RL^n = R(L\circ L^{n-1}) = (RL)\circ L^{n-1} \subset L\circ L^{n-1} = L^n$$

PROPOSITION 2.15. Suppose R satisjes S1. For any right ideal L of R and any positive integers n and m

$$L^n L^m = L^{n+m}$$

Proof. This is clear by Definition 2.4 (iii), S1, and Proposition 2.14

PROPOSITION 2.16. Suppose R satisfies S1. If L_1 and L_2 are nilpotent right ideals of R, then $L_1 + L_2$ is a nilpotent right ideal of R.

Proof. Suppose that L_1, is a nilpotent right ideal of R with nilpotent exponent n and that L_2 is a nilpotent right ideal of R with nilpotent exponent m. Clearly, $L_1 + L_2$ is a right ideal of R.

By Proposition 2.15, to show that $L_1 + L_2$ is nilpotent, it suffices to show that any product

$$(...(((a_1 a_2)a_3)a_4)...)a_{m+n} \qquad (2.3)$$

is zero, where each a_i belongs to L_1 or L_2.

Without loss of generality, we may assume that we have at least n elements from L_1 in the product (2.3). Since ba_i belongs to the same right ideal as a_i, we may also assume that $a_1 \in L_1$. Suppose we have written the product (2.3) in the form

$$(...(((b_1 b_2)b_3)b_4)...)b_t, \qquad (2.4)$$

where $b_1 \in L^k{}_1$ and $b_i \in L_1$ or $b_i \in L_2$ for $2 \le i \le t$ with at least $n-k$ of the $b_i, i > 1$, belonging to L_1.

If $b_2 \in L_1$, then $b_1 b_2 \in L_1^{k+1}$ and

$$(...(((b_1 b_2)b_3)b_4)...)b_t = (...(((c_1 c_2)c_3)c_4)...)c_{i-1}$$

where $c_1 = b_1 b_2 \in L_1^{k+1}$, $c_i = b_{i+1}$ for $i > 1$ and at least $n-(k+1)$ of the $c_i, i > 1$, belong to L_1.

Suppose $b_2 \notin L_1$. Let $q > 2$ be the least integer such that $b_q \in L_1$. S1 now gives

$$\begin{aligned}(...(((b_1 b_2)b_3)b_4)...)b_t &= (...(((b_{11}(b_{21}b_3))b_4)...)b_t \\ &= (...(((c_1 c_2)c_3)c_4)...)c_{t-1} \qquad (2.5)\end{aligned}$$

Where $c_1 = b_{11} \in L_1^k, c_2 = b_{21}b_3, c_i = b_{i+1}$ for $i > 2$ and $c_{q-1} \in L_1$. Hence, by induction we may now assume that $b_3 \in L_1$. Then from (2.5) we have:

$$(...(((b_1 b_2)b_3)b_4)...)b_t = (...(((c_1 c_2)c_3)c_4)...)c_{t-1} = (...(((d_1 d_2)d_3)d_4)...)d_{t-2},$$

where $d_1 = c_1c_2 = c_1(b_{21}b_3) \in L_1^{k+1}, d_i = c_{i+1}$ for $i > 1$ and at least $n-(k+1)$ of the $d_i, i > 1$, belong to L_1.

By induction, any product of the form (2.3) with n many $a_i \in L_1$, belongs to $L_1{}^n$ or $L_1{}^nL_2$. But

$$L_1{}^n = L_1{}^nL_2 = \{0\}.$$

PROPOSITION 2.17. Suppose R satisfies S1 and L is a right ideal of R. Then $R \circ L$ is an ideal of R. If L is nilpotent, $R \circ L$ is nilpotent.

Proof. Suppose $x \in L$ and $r, s \in R$. Then, by S1, there are $\overline{x} \in L$ and $\overline{r} \in R$ such that $(rx)s = \overline{r}(\overline{x}s) \in RL$. Also by S1, there is a $t \in R$ and a $y \in L$ such that $s(rx) = r(ty) \in RL$. Thus, $R \circ L$ is an ideal of R.

The remainder of the proof is similar to the reassociation argument found in the proof of Proposition 2.16.

DEFINITION 2.18. Suppose R satisfies S1 and A.C.C. By Proposition 2.16 and Proposition 2.17, R has a unique maximal nilpotent ideal N which we call the radical of R and denote by rad R. If rad $R = \{0\}$ we say that R is semi-simple.

PROPOSITION 2.19. Suppose R satisfies S1, A.C.C. and has an identity. Then $\{0\} \neq R / radR$ and $R / radR$ is semi-simple.

Proof. R has a maximal nilpotent right ideal L. By Proposition 2.17 $L \subset R \circ L$ where $R \circ L$ is a nilpotent ideal of R. Thus, $L \subset N$. If L_1 / N is a nilpotent right ideal of R / N, then L_1 is a right ideal of R such that $L_1{}^n \subset N$, for some n. Then, $(L_1{}^n)^m = L_1{}^{nm} \subset N^m = \{0\}$, where m is the nilpotent exponent of N. Thus L_1 is nilpotent and $L_1 \subset N$.

We can obtain somewhat sharper results for radical theory under S3.

PROPOSITION 2.20. Suppose R satisfies S3, D.C.C. and has an identity. Then a nil right ideal of R must be nilpotent.

Proof. Suppose L is a nil but non-nilpotent right ideal of R. Without loss of generality we may assume that L is a minimal nil but non-nilpotent right ideal of R.

By Proposition 2.15, if L^2 is nilpotent, L is nilpotent. Hence $L^2 \subset L$ gives that $L^2 = L$.

Let $\chi = \{M : M \text{ is a right ideal of } R \text{ with } M \subseteq L \text{ and } ML \neq \{0\}\}$. $L \in \chi$ so that $\chi \neq \varnothing$. Let M be a minimal right ideal in χ.

$LM \neq \{0\}$ so that there is a $u \in M$ with $uL \neq \{0\}$. By Proposition 2.13, uL is a right ideal of R and $uL \subset ML \subset M \subset L$. $\because$ $(uL)L = uL^2 = uL$, by S3 (or 1). Hence, $uL = M$, by the minimality of M.

Hence there is an $e \in L$ such that $eu = u \neq 0$. Since $e \in L$ and L is nil, there is a positive integer n such that $e^n = 0$. But for $q \geq 1$,

$$e^q(e-1) = e^{q+1} - e^q \in \ell - ann(u),$$

by Proposition 2.13. Then $0 = e^n u = e^{n-1} u = ... = eu = u \neq 0$ is a contradiction. Thus every nil right ideal of R is nilpotent.

DEFINITION 2.21. If R satisfies S3, D.C.C. and has an identity element, we define the radical of R, denoted rad R, to be the sum of all its nilpotent right ideals. If rad $R = \{0\}$, we say that R is semi-simple.

PROPOSITION 2.22. If R satisfies S3, D.C.C., and has an identity, then rad R is a nilpotent ideal of R and $R / rad\ R$ is semi-simple.

Proof. By Proposition 2.20, rad R is a nilpotent right ideal of R. By Proposition 2.17, $R \circ (rad\ R)$ is a nilpotent ideal of R so we have:

$$R \circ (rad\ R) \subset rad\ R \subset R \circ (rad\ R)$$

so that rad R is an ideal of R. Clearly $R / rad\ R$ is semi-simple.

Note. If R satisfies S3, A.C.C., D.C.C., and has an identity, then Proposition 2.22 shows that the two definitions of rad R, Definition 2.19 and Definition 2.21, agree.

2 4. Semi-Simple Theory

DEFINITION 2.23. Let R be a ring satisfying S1 and D.C.C. Suppose also that R has no non-zero nilpotent right ideals. Defined by

$$\chi = \{L : L \text{ is a minimal right ideal of } R\}.$$

If $L_1, L_2 \in \chi$ we write $L_1 \approx L_2$ if there is an $x \in L_2$ such that $L_2 = xL_1$.

PROPOSITION 2.24. Suppose we have R and χ as in DeJinition 2.23. Then $\approx$ is an equivalence relation on χ. Furthermore, for $L_1, L_2 \in \chi, L_1 L_2 \neq \{0\}$ if and only if $L_1 L_2 = L_2$, if and only if $L_1 \approx L_2$.

The proof is straightforward, using the semi-simplicity of R and the minimality of the right ideals belonging to χ.

THEOREM 2.25. Suppose R is a finite-dimensional, semi-simple algebra under S1. Then we may write $R = L \oplus M$, where $L = L_1 \oplus ... \oplus L_n$, a sum of minimal right ideals of R, and L is an ideal of R such that R contains no minimal right ideal not already contained in L. Hence, R has a composition series of right ideals.

Proof. First suppose that I is a right ideal of R and L_1 is a minimal right ideal of R. Then either $L_1 \subset I$ or $L_1 \cap I = \{0\}$. Suppose $0 \neq x \in L_1 \cap I$. Then $x \in xR \subset L_1$ and $xR \subset I$. By the minimality of L_1, $L_1 = xR \subset I$.

By the argument above, suppose we have $R = L_1 \oplus L_2 \oplus ... \oplus L_n \oplus M$ where the L_i are minimal right ideals of R and there are no minimal right ideals of R not contained in L.

Suppose that L is not an ideal of R. Then, since $R \circ L$ is an ideal of R, there is an $x \in R$ such that $xL_i \not\subset L$ for some i. xL_i must contain a minimal right ideal, say I. Now, $I = I^2 \subset I(xL_i) = x(IL_i)$ so that $I \approx L_i$ and $L_i \approx I$, by Proposition 2.24. But $\dim I \le \dim(xL_i) \le \dim L_i$ and, since for some $y \in L_i$, $L_i = yI$, $\dim L_i \le \dim(yI) \le \dim I$. Hence $I = xL_i$ and xL_i is a minimal left ideal. But $xL_i \not\subset L$ is a contradiction. Thus, L is an ideal of R.

Example 2.11, Case 2 shows that a semi-simple ring with identity under S2 need not be a direct sum of simple ideals. The difficulty here is that the sum of all minimal right ideals of R may not give all of R; see Theorem 2.25. However, we will see that this does not occur under S3.

Henceforth in Section 2.4 we will assume that R is a semi-simple ring with identity 1 under S3 and D.C.C.

PROPOSITION 2.26. Suppose that L is a minimal right ideal of R. Then L contains a non-zero idempotent e. Furthermore, $L = e\mathrm{R}$ and $R = e\mathrm{R} \oplus (1-e)R$, where the sum is additively direct.

Proof. (a) Since R contains no nilpotent right ideals, $L^2 = L \neq \{0\}$. Then there is a $u \in L$ such that $uL = L$. Hence there is an $e \in L$ such that $ue = u$. Then $e(e-1) = e^2 - e \in L \cap \ell - ann(u)$. But $e \notin L \cap \ell - ann(u)$, so by Proposition 2.13, $L \cap \ell - ann(u) = \{0\}$ and $e^2 = e$. Clearly $e\mathrm{R} = L$.

(b) $R = e\mathrm{R} + (1-e)R$ since $x = ex + (1-e)x$. Claim: $(1-e)R = \{x \in R : ex = 0\}$.

If $x \in (1-e)R$, then $x = (1-e)y$ for some $y \in R$. Then,

$$ex = e((1-e)y) = ((1-e)e)y = 0$$

Conversely, if $ex = 0$, then

$$x = (1-e)x \in (1-e)R.$$

Now $e\mathrm{R} \cap (1-e)R \underset{\neq}{\subset} e\mathrm{R}$ since $e \in e\mathrm{R}$ and $e^2 = e \neq 0$. But $e\mathrm{R} = L$ so that, by the minimal of

$$L, e\mathrm{R} \cap (1-e)R = \{0\}.$$

PROPOSITION 2.27. Every minimal right ideal of R is a direct summand of any containing left ideal. Also, we may write $R = \mathrm{e}_1\mathrm{R} \oplus \mathrm{e}_2\mathrm{R} \oplus ... \oplus \mathrm{e}_n\mathrm{R}$, where the $\mathrm{e}_i\mathrm{R}$ are minimal right ideals and the e_i are non-zero idempotents.

Proof. The proof is as in the associative case using Proposition 2.26.

Remark 2.28, We have

$$R = L_1 \oplus L_2 \oplus ... \oplus L_n.$$

Write $1 = f_1 + f_2 + ... + f_n$, where $f_i \in L_i$. $f_i = f_i f_1 + f_i f_2 + ... + f_i f_n$ so that $f_i f_j = \delta_{ij} f_j$. Also, $L_i = f_i R$ so we may replace every f_i by e_i and use $1 = e_1 + e_2 + ... + e_n$.

By Proposition 2.24, there are finitely many equivalence classes under $\approx$ in R, say $k(\leq n)$ many. For each j with $1 \leq j \leq k$ pick a representative I_j for the k disjoint equivalence classes. Define $B_j = \sum_{L_i \approx I_j} L_i$. We may write $R = B_1 \oplus B_2 \oplus ... \oplus B_k$, where each B_i is a simple component for R . Now, as in the associative case, we obtain:

THEOREM 2.29. If R is a semi-simple ring with identity under S3 and D.C.C., then R is a finite-ring direct sum of simple rings each satisfying S3 with descending chain condition and identity.

CONCLUSION

Non-associative rings extend the landscape of algebra by showing that meaningful and highly structured systems can exist even when one of the most familiar laws—associativity of multiplication—is removed. While this absence introduces additional complexity, it also reveals deeper layers of algebraic behavior, where identities such as alternativity, flexibility, or the Lie bracket take on central roles in shaping structure and guiding computation.

Throughout their study, non-associative rings demonstrate that algebra is not confined to rigid axioms but is instead a flexible framework capable of adapting to diverse mathematical and physical contexts. From Lie theory and Jordan algebra theory to applications in quantum mechanics and geometry, these structures provide essential tools for modeling symmetries, observables, and non-classical interactions.

In conclusion, non-associative rings highlight both the challenges and opportunities that arise when foundational assumptions are relaxed. Their rich theory not only generalizes classical algebra but also opens pathways to new areas of research, offering insight into systems where associativity is either too restrictive or fundamentally absent.

COMPETING INTEREST

Author have declared that no competing interests exist.

REFERENCES

1. Baer, R. (1952). Linear algebra & Projective geometry. (1st ed.). New York: Academic Press Inc.
2. Bracic, J. (2001). Representations & Derivations of modules. Irish Mathematical Society Bulletin, 47(2001), 27-39.
3. Rim, S.H. (1987). Extensions of high anti-derivations to modules of quotients. Journal of the Korean Mathematical Society,24(1), 25-31.
4. Osborn, H. (1968). Module of differentials II. Mathematische Annalen, 175, 146-158.
5. Matsumura, H. (1986). Commutative ring theory. (1st ed.). Cambridge, UK: Cambridge University Press.

6. Karakuş, A. (2021). An approximation to second exterior derivation of high order universal modules, Algebra Letters, 1(2021), 1-13.
7. Hart, R. (1996). Higher derivations & Universal differentials operators. Journal of Algebra, 0253,184,175-181. https://doi.org/10.1006/jabr.1996.0253.